INFO I201  
Homework 4  
A few solutions

1. Check the validity of the following argument. Either prove that it is valid or show by a counterexample that it is invalid.

\[
\begin{align*}
B & \rightarrow (\neg S \lor M) \\
S \lor M \\
M & \rightarrow H \\
B & \rightarrow H
\end{align*}
\]

Clearly all the paths are closed. I didn't completely work out the tree ($\times_1$ and $\times_2$), but you can see the paths are closed there too.

2. Check the following arguments for validity? In each case, either prove that it is valid or show by a counterexample that it is invalid.

(a) \[
\begin{align*}
A \\
A & \rightarrow B \\
B & \rightarrow C \\
\hline
C
\end{align*}
\]

We can reason through this, since we have experience in formal proofs. Clearly $A$ must be true, and thus $A \rightarrow B$ is true only when $B$ is true. In turn, to make $B \rightarrow C$ true, $C$ must be true. Hence, $C$ is true.
(b) \[\begin{align*}
A & \rightarrow (B \land C) \\
\neg B & \rightarrow \neg A \\
\neg C & \\
\hline
A & \rightarrow (B \lor C)
\end{align*}\]

All paths are closed, so the argument is valid.

3. There is an island in Pacific called the Island of Knights and Knaves. There are two groups of inhabitants on this island, namely Knights who always tell the truth and Knaves who always lie. Can you solve the following puzzle:

We have three people \(A\), \(B\), and \(C\) on the Island of Knights and Knaves. Suppose \(A\) and \(B\) say the following:

A: All of us are knaves.

B: Exactly one of us is a knave.

Can it be determined what \(B\) is? Can it be determined what \(C\) is?.

What we’re looking for is a model that is consistent with the statements. We must assign to one of two distinct groups \(A, B, C\) to either knights or knaves. The knights always tell the truth, and the knaves always lie. An easy way to discover a model in such a modest setting is by enumeration. Let \(I(\cdot)\) mean the inhabitant is a knight, and \(V(\cdot)\) mean the inhabitant is a knave. The possibilities are:
We investigate the statements in light of each model. $A$ is stating that $V_A = \{A, B, C\}$ and $B$ one of either $V_B = \{A\}$ or $V_B = \{B\}$ or $V_B = \{C\}$. What we have to do is check the validity of $A$ and $B$’s statements. I’ve subscripted the set names to keep track of who is making the statement.

$m_1$ This cannot be a model since $A$ is a knight and cannot lie–similarly for $B$.
$m_2$ This cannot be a model (see above).
$m_3$ $A$ cannot lie, and does. $B$ must lie, but tells the truth. This cannot be a model.
$m_4$ This can be true. $A$ is lying about the number of knaves and $B$ is telling the truth about the number of knaves.
$m_5$ This can be true. $A$ is lying about the number of knaves and $B$ is lying about the number of knaves.
$m_6$ $A$ can lie, but $B$ would be lying as a knight–this cannot work.
$m_7$ $A$ cannot lie, so this cannot work.
$m_8$ $A$ would be telling the truth, something that can’t be–so this cannot work.

So, models $m_4$ and $m_5$ both work. While we cannot determine, $B$, we can determine the $C$ must be a knight.

4. On the Island of Knights and Knaves, three inhabitants $A, B, C$ are being interviewed. $A$ and $B$ make the following statements:

$A$ : $B$ is a knight.

$B$ : If $A$ is a knight so is $C$.

Can it be determined what any of $A$, $B$, $C$ are?

5. Again three people $A, B$ and $C$. $A$ says, “$B$ and $C$ are of the same type.” Someone then asks $C$, “Are $A$ and $B$ of the same type?” What does $C$ answer?

6. Check the following specification for consistency:

The system is in multiuser state if and only if it is operating normally. If the system is operating normally, the kernel is functioning. The kernel is not functioning or the system is in interrupt mode. If the system is not in

<table>
<thead>
<tr>
<th>Model</th>
<th>$I$</th>
<th>$V$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m_1$</td>
<td>$A, B, C$</td>
<td></td>
</tr>
<tr>
<td>$m_2$</td>
<td>$A, B$</td>
<td>$C$</td>
</tr>
<tr>
<td>$m_3$</td>
<td>$A, C$</td>
<td>$B$</td>
</tr>
<tr>
<td>$m_4$</td>
<td>$C, B$</td>
<td>$A$</td>
</tr>
<tr>
<td>$m_5$</td>
<td>$C$</td>
<td>$B, A$</td>
</tr>
<tr>
<td>$m_6$</td>
<td>$B$</td>
<td>$C, A$</td>
</tr>
<tr>
<td>$m_7$</td>
<td>$A$</td>
<td>$B, C$</td>
</tr>
<tr>
<td>$m_8$</td>
<td></td>
<td>$A, B, C$</td>
</tr>
</tbody>
</table>
multiuser state, then it is in interrupt mode. The system is not in interrupt mode.

7. We have three inhabitants $A$, $B$, and $C$ on the Island of Knights and Knaves. $A$ and $B$ make the following statements:

$A$: “I am a knave but $B$ is not a knave.”
$B$: “All of us are knights.”

Again we need to find models that are consistent with the statements. We can enumerate the models as we did before.

<table>
<thead>
<tr>
<th>Model $m$</th>
<th>$I$</th>
<th>$V$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m_1$</td>
<td>$A, B, C$</td>
<td></td>
</tr>
<tr>
<td>$m_2$</td>
<td>$A, B$</td>
<td>$C$</td>
</tr>
<tr>
<td>$m_3$</td>
<td>$A, C$</td>
<td>$B$</td>
</tr>
<tr>
<td>$m_4$</td>
<td>$C, B$</td>
<td>$A$</td>
</tr>
<tr>
<td>$m_5$</td>
<td>$C$</td>
<td>$B, A$</td>
</tr>
<tr>
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<td>$B$</td>
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</tr>
<tr>
<td>$m_8$</td>
<td>$A, B, C$</td>
<td></td>
</tr>
</tbody>
</table>

$A$ is saying he is a knave and $B$ is a knight. $B$ says all are knights.

- $m_1$ $A$ cannot lie, so this isn’t a model.
- $m_2$ Same as above.
- $m_3$ $A$ cannot lie.
- $m_4$ $A$ is telling the truth and can’t be—so this isn’t a model.
- $m_5$ $A$ and $B$ are both lying.
- $m_6$ $A$ is telling the truth and can’t be—so this isn’t a model.
- $m_7$ $A$ can’t be telling the truth—so this isn’t a model.
- $m_8$ $A$ and $B$ are both lying.

$C$ can be either a knight or a knave, but both $A$ and $B$ are knaves.

Can you determine what $A$, $B$ and $C$ are?

8. In each case translate the given statement into propositional logic using the letters given.

(i) It snows, only if the wind blows from the Northeast or weather is very cold. (S,W,C).

(ii) For you to win the contest it is necessary that you have the winning ticket. (W,T)

(iii) I will remember to send you the address if and only if you send me an e-mail message. (A,M)
(iv) Grizzly bears have not been seen in the area and hiking on the trail is safe, given that berries are ripe along the trail. (G,H,B)

(v) I will give you my car if you want me to, but only if you do not use it in another bank robbery. (C,W,B)

(vi) If the president knew about the diversion of funds to the rebels, then he deliberately violated the law and does not deserve to be president; but if he did not know about the diversion of funds, then he did not have control of his subordinates and does not deserve to be president. (K,V,D,C)

(vii) Doubts arise when knowledge is not certain, but if knowledge is certain then either it is trivial and not worth doubting, or if not trivial it is at best defectively justifiable. (D,K,T,W,J)

9. Are these system specifications consistent? If yes, give truth values for the atomic propositions involved that make all statements true.

“Users cannot access the file system whenever the system software is being upgraded. Users can access the file only if they can save new files. If users can save new files, then the system software is not being upgraded.”

10. Consider the following argument:

“The Republicans will lose power only if the dollar drops or the debt increases. If the dollar drops and the debt increases, then the Republicans will lose power. The debt does not increase if the dollar does not drop. The dollar drops. Therefore the Republicans will lose power.”

Is this argument valid? If yes, give a proof, else give a counterexample.

11. Consider the following argument:

“People are naturally good only if the law is unnecessary. People are naturally evil only if the law is ineffective. Unless people are naturally good, people are naturally evil. So, either the law is unnecessary or ineffective.

Is this argument valid? If yes, give a proof, else give a counterexample.

12. Are the following statements consistent?

“When the mixture begins to bubble, there will be a delicious aroma of vanilla beans, if the ingredients are unspoiled. As the mixture begins to cool, there will be a faint red striation across the surface of the liquid. Either the mixture cools or there will be a delicious aroma of vanilla beans.” (use B, A, S, C, R).

13. State the converse and the contrapositive of each of the following statements:

(a) If it snows today, I will ski tomorrow.
(b) I come to class whenever there is going to be a quiz.
(c) When I stay up late, it is necessary that I sleep until noon.
(d) I go to the beach whenever it is a sunny summer day.

14. Give natural deduction proofs for the following sequents.

(a) \( A, A \rightarrow B, B \rightarrow C \vdash A \land C \)
(b) \( A \rightarrow B \vdash A \rightarrow B \)
(c) \( \vdash A \rightarrow (B \rightarrow A) \)
(d) \( (A \land B) \rightarrow (C \land D), A \vdash B \rightarrow D \)
(e) \( A \rightarrow (B \rightarrow C) \vdash (A \land B) \rightarrow C \)
(f) \( A \land \neg B \vdash \neg(A \land B) \)
(g) \( (A \land B) \rightarrow C \vdash A \rightarrow (B \rightarrow C) \)
(h) \( A \rightarrow B \vdash \neg B \rightarrow \neg A \)
(i) \( A \rightarrow B, \neg B \vdash \neg A \)
(j) \( A, A \rightarrow C, C \rightarrow (D \land E) \vdash E \)

15. Use set builder notation to give a description of each of these sets. That is you need to find a property that describes the elements in the set.

- \( \{0, 3, 6, 9, 12\} \)
- \( \{-3, -2, -1, 0, 1, 2\} \)
- \( \{1, 3, 5, 7, \cdots \} \)

16. Suppose that \( A = \{2, 4, 6\}, B = \{2, 6\}, C = \{4, 6\}, \) and \( D = \{4, 6, 8\} \). Determine which of these sets are subsets of which other of these sets.

17. Determine whether each of these statements is true or false:

- \( 0 \in \emptyset \)
- \( \{0\} \subseteq \emptyset \)
- \( \emptyset \in \{0\} \)
- \( \{\emptyset\} \in \{\emptyset\} \)
- \( \{\emptyset\} \subseteq \{\{\emptyset\}, \{\emptyset\} \} \)
- \( \emptyset \in \{\emptyset, \{\emptyset\}\} \)

18. What is the cardinality of each of these sets?

- \( \emptyset \)
- \( \{\emptyset\} \)
- \( \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\} \)
19. Determine whether each of these sets is the power set of a set.
   • \( \emptyset \)
   • \( \{\emptyset, \{a\}\} \)
   • \( \{\emptyset, \{a\}, \{\emptyset, a\}\} \)

20. Suppose that \( A \times B = \emptyset \), where \( A \) and \( B \) are sets. What can you conclude? Explain.

21. Let \( A = \{a, b, c\} \), \( B = \{x, y\} \) and \( C = \{0, 1\} \). Find
   • \( A \times B \times C \)
   • \( C \times B \times A \)
   • \( C \times A \times B \)
   • \( B \times B \times B \)

22. Let \( A = \{a, b, c, d, e\} \) and \( B = \{a, b, c, d, e, f, g, h\} \). Find
   • \( A \cup B \)
   • \( A \cap B \)
   • \( A - B \)
   • \( B - A \)